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On the nonisospectral Kadomtsev–Petviashvili equation

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Abstract

In this paper, we first present the Grammian determinant solutions to the nonisospectral Kadomtsev–Petviashvili (KP) equation. Then, by using the Pfaffianization procedure of Hirota and Ohta, an integrable coupled system is generated. Moreover, Gramm-type Pfaffian solutions to the Pfaffianized system are proposed.

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1. Introduction

In the early 1990s, Hirota and Ohta [1, 2] developed a procedure for generalizing equations from the Kadomtsev–Petviashvili (KP) hierarchy to produce coupled systems of equations, which we now call Pfaffianization. These Pfaffianized equations appear as coupled systems of the original equations and have soliton solutions expressed by Pfaffians. Such a procedure has been successfully applied to the DS equations [3], the discrete KP equation [4], the self-dual Yang–Mills equation [5], the two-dimensional Toda lattice [6], the semi-discrete Toda equation [7], the differential-difference KP equation [8], etc.

In [9], Wronskian solutions of the nonisospectral KP equation [10]

$$4u_t + y\left(u_{xxx} + 6uu_x + 3\partial_x^{-1}u_{yy}\right) + 2xu_y + 4\partial_x^{-1}u_y = 0 \tag{1}$$

are derived by the Hirota method and the Wronskian technique. The solutions of the bilinear nonisospectral KP equation are expressed in Wronskian determinants. Since there are both Wronskian determinant and Grammian determinant solutions for the KP equation [11], we expect that there also exist Gramm-type expressions for (1). On the other hand, all the Pfaffianization procedures above are applied to isospectral systems. Hence it would be very interesting to consider Pfaffianization for nonisospectral systems.

This paper is organized as follows. In section 2, we give the Grammian solution to the nonisospectral KP equation (1). In section 3, we apply the Pfaffianization procedure to (1). As a result, a coupled system is derived with the help of the quadratic Pfaffian identities. The Gramm-type Pfaffian solutions to the coupled system are then presented.

2. Grammian solution for the nonisospectral KP equation

Through the dependent variable transformation $u = 2(\ln f)_{xx}$, the nonisospectral KP equation (1) can be transformed into the bilinear form

$$4D_xD_tf\cdot f+y\big(D_x^4f\cdot f+3D_y^2f\cdot f\big)+2xD_xD_yf\cdot f+4f_yf=0, \tag{2}$$

where the bilinear operators D_x^m and D_y^n are defined by [1, 2]

$$D_x^mD_y^na\cdot b\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^n a(x,y)b(x',y')\bigg|_{x'=x,y'=y}.$$

We know that the bilinear nonisospectral KP equation (2) has a solution expressed in the Wronskian form [9]

$$f=W(\phi_1,\phi_2,\ldots,\phi_N)=\begin{vmatrix}\phi_1 & \partial\phi_1 & \cdots & \partial^{N-1}\phi_1\\ \phi_2 & \partial\phi_2 & \cdots & \partial^{N-1}\phi_2\\ \cdots & \cdots & \cdots & \cdots\\ \phi_N & \partial\phi_N & \cdots & \partial^{N-1}\phi_N\end{vmatrix}, \tag{3}$$

where the elements $\phi_j (j = 1, 2, \ldots, N)$ satisfy the differential relations

$$\phi_{jy}=\phi_{jxx},\qquad \phi_{jt}=-y\phi_{jxxx}-\tfrac{1}{2}x\phi_{jxx}+\tfrac{1}{2}(N-1)\phi_{jx}. \tag{4}$$

Besides the Wronskian solution, equation (2) possesses a solution f expressed in the Gramm determinant

$$f=\det\left|c_{ij}+\int^x f_ig_j\,\mathrm{d}t\right|_{1\leqslant i,j\leqslant N},\qquad c_{ij}=\text{const}, \tag{5}$$

where f_i and g_j satisfy the linear differential equations

$$f_{iy}=f_{ixx},\qquad g_{jy}=-g_{jxx}, \tag{6}$$

$$f_{it}=-yf_{ixxx}-\tfrac{1}{2}xf_{ixx}-\tfrac{1}{2}f_{ix}, \tag{7}$$

$$g_{jt}=-yg_{jxxx}+\tfrac{1}{2}xg_{jxx}+\tfrac{1}{2}g_{jx}. \tag{8}$$

It is known that any determinant can be expressed by a Pfaffian. We rewrite f as

$$f=(1,2,\ldots,N,N^*,\ldots,2^*,1^*), \tag{9}$$

$$(i,j^*)=c_{ij}+\int^x f_ig_j\,\mathrm{d}x,\qquad (i,j)=(i^*,j^*)=0. \tag{10}$$

In order to prove that f satisfies equation (2), we introduce Pfaffians defined by

$$(d_n,j^*)=\frac{\partial^n}{\partial x^n}g_j,\qquad (d_m,d_n^*)=0, \tag{11}$$

$$(d_n^*,i)=\frac{\partial^n}{\partial x^n}f_i,\qquad (d_m^*,i^*)=(d_n,i)=0. \tag{12}$$

By virtue of these Pfaffians and equations (6)–(8), we come up with the following differential formulae for f:

$$f_x = (d_0, d_0^*, \cdot), \qquad f_{xx} = (d_0, d_1^*, \cdot) + (d_1, d_0^*, \cdot), \tag{13}$$

$$f_y = (d_0, d_1^*, \cdot) - (d_1, d_0^*, \cdot), \tag{14}$$

$$f_{xxx} = (d_0, d_2^*, \cdot) + (d_2, d_0^*, \cdot) + 2(d_1, d_1^*, \cdot), \tag{15}$$

$$f_t = (-y)[(d_2, d_0^*, \cdot) - (d_1, d_1^*, \cdot) + (d_0, d_2^*, \cdot)] - \tfrac{1}{2}x[d_0, d_1^*, \cdot) - (d_1, d_0^*, \cdot)], \tag{16}$$

$$f_{xxxx} = (d_3, d_0^*, \cdot) + 3(d_2, d_1^*, \cdot) + 2(d_0, d_0^*, d_1, d_1^*, \cdot) + 3(d_1, d_2^*, \cdot) + (d_0, d_3^*, \cdot), \tag{17}$$

$$f_{yy} = (d_0, d_3^*, \cdot) - (d_2, d_1^*, \cdot) + 2(d_0, d_0^*, d_1, d_1^*, \cdot) + (d_3, d_0^*, \cdot) - (d_1, d_2^*, \cdot), \tag{18}$$

$$f_{tx} = (-y)[(d_3, d_0^*, \cdot) - (d_0, d_0^*, \cdot) + (d_0, d_3^*, \cdot)] - \tfrac{1}{2}[(d_0, d_1^*, \cdot) - (d_1, d_0^*, \cdot)] \\ - \tfrac{1}{2}x[(d_0, d_2^*, \cdot) - (d_2, d_0^*, \cdot)]. \tag{19}$$

Here we have denoted $f = (1, \ldots, N, N^*, \ldots, 1^*) = (\cdot)$.

Substituting the above Pfaffians into equation (2), after some calculations, we get the Jacobi identities

$$24y\{(d_0, d_0^*, d_1, d_1^*, \cdot)(\cdot) - (d_0, d_0^*, \cdot)(d_1, d_1^*, \cdot) + (d_0, d_1^*, \cdot)(d_1, d_0^*, \cdot)\} \equiv 0. \tag{20}$$

Thus we have proved that f given by (5) is the Grammian solution for equation (2).

3. The coupled system for the nonisospectral KP equation

In this section, we will first apply the Pfaffianization procedure to equation (2). Then we will present the Gramm-type Pfaffian solution to its Pfaffianized system. For this purpose, we need the Pfaffian identities

$$(a_1, a_2, \ldots, a_{N-2}, \alpha, \beta, \gamma, \delta)(a_1, a_2, \ldots, a_{N-2}) \\ - (a_1, a_2, \ldots, a_{N-2}, \alpha, \beta)(a_1, a_2, \ldots, a_{N-2}, \gamma, \delta) \\ + (a_1, a_2, \ldots, a_{N-2}, \alpha, \gamma)(a_1, a_2, \ldots, a_{N-2}, \beta, \delta) \\ - (a_1, a_2, \ldots, a_{N-2}, \alpha, \delta)(a_1, a_2, \ldots, a_{N-2}, \beta, \gamma) = 0 \tag{21}$$

and

$$(a_1, a_2, \ldots, a_{N-1}, \alpha, \beta, \gamma)(a_1, a_2, \ldots, a_{N-1}, \delta) \\ - (a_1, a_2, \ldots, a_{N-1}, \alpha, \beta, \delta)(a_1, a_2, \ldots, a_{N-1}, \gamma) \\ + (a_1, a_2, \ldots, a_{N-1}, \alpha, \gamma, \delta)(a_1, a_2, \ldots, a_{N-1}, \beta) \\ - (a_1, a_2, \ldots, a_{N-1}, \beta, \gamma, \delta)(a_1, a_2, \ldots, a_{N-1}, , \alpha) = 0. \tag{22}$$

3.1. Pfaffianization of the bilinear nonisospectral KP equation (2)

In order to Pfaffianize the nonisospectral KP equation (2), we require a Pfaffian with elements satisfying the Pfaffianized form of the dispersion relation (4). In much the same way as in [1], we take $f = (1, 2, \ldots, 2N)$ and the entries in our Pfaffian are chosen to satisfy

$$\frac{\partial}{\partial x}(i, j) = (i + 1, j) + (i, j + 1),$$
$$\frac{\partial}{\partial y}(i, j) = (i + 2, j) + (i, j + 2),$$

$$\frac{\partial}{\partial t}(i,j) = -y[(i+3,j)+(i,j+3)] - \tfrac{1}{2}x[(i+2,j)+(i,j+2)] \\ + \tfrac{1}{2}(2N-i)(i+1,j) + \tfrac{1}{2}(2N-j)(i,j+1). \tag{23}$$

As an example, we can take the entry

$$(i,j) = \sum_{k=1}^{M}\left[\Phi_k^{(i)}\Psi_k^{(j)} - \Phi_k^{(j)}\Psi_k^{(i)}\right], \tag{24}$$

where M is an arbitrary natural number and $\Phi_k^{(i)}$ and $\Psi_k^{(i)}$ stand for i-times differentials with respect to x. Φ_k and Ψ_k satisfy the following differential rules:

$$\begin{aligned} &\Phi_{ky} = \Phi_{kxx}, \qquad \Psi_{ky} = \Psi_{kxx}, \\ &\Phi_{kt} = -y\Phi_{kxxx} - \tfrac{1}{2}x\Phi_{kxx} + N\Phi_{kx}, \\ &\Psi_{kt} = -y\Psi_{kxxx} - \tfrac{1}{2}x\Psi_{kxx} + N\Psi_{kx}. \end{aligned} \tag{25}$$

We can calculate that

$$\begin{aligned} f_x &= (1,2,\ldots,2N-1,2N+1), \\ f_{xx} &= (1.2,\ldots,2N-1,2N+2) + (1,2,\ldots,2N-2,2N,2N+1), \\ f_{xxx} &= (1,2,\ldots,2N-1,2N+3) + 2(1,2,\ldots,2N-2,2N,2N+2) \\ &\quad + (1,\ldots,2N-3,2N-1,2N,2N+1), \\ f_{xxxx} &= (1,2,\ldots,2N-1,2N+4) + 3(1,2,\ldots,2N-2,2N,2N+3) \\ &\quad + 3(1,2,\ldots,2N-3,2N-1,2N,2N+2) \\ &\quad + 2(1,2,\ldots,2N-2,2N+1,2N+2) \\ &\quad + (1,2,\ldots,2N-4,2N-2,2N-1,2N,2N+1), \\ f_y &= (1,2,\ldots,2N-1,2N+2) - (1,2,\ldots,2N-2,2N,2N+1), \\ f_t &= -y[(1,2,\ldots,2N-1,2N+3) + (1,2,\ldots,2N-2,2N+2,2N) \\ &\quad + (1,2,\ldots,2N-3,2N+1,2N-1,2N)] \\ &\quad - \tfrac{1}{2}x[(1,2,\ldots,2N-1,2N+2) - (1,2,\ldots,2N-2,2N,2N+1)], \end{aligned} \tag{26}$$

$$\begin{aligned} f_{yx} &= (1,2,\ldots,2N-1,2N+3) - (1,2,\ldots,2N-3,2N-1,2N,2N+1), \\ f_{yy} &= 2(1,2,\ldots,2N-2,2N+1,2N+2) + (1,2,\ldots,2N-1,2N+4) \\ &\quad - (1,2,\ldots,2N-3,2N-1,2N,2N+2) - (1,2,\ldots,2N-2,2N,2N+3) \\ &\quad + (1,2,\ldots,2N-4,2N-2,2N-1,2N,2N+1), \\ f_{tx} &= -y[(1,2,\ldots,2N-1,2N+4) - (1,2,\ldots,2N-2,2N+1,2N+2) \\ &\quad + (1,2,\ldots,2N-4,2N-2,2N-1,2N,2N+1)] \\ &\quad - \tfrac{1}{2}[(1,2,\ldots,2N-1,2N+2) - (1,2,\ldots,2N-2,2N,2N+1)] \\ &\quad - \tfrac{1}{2}x[(1,2,\ldots,2N-1,2N+3) - (1,2,\ldots,2N-3,2N-1,2N,2N+1)]. \end{aligned} \tag{27}$$

For simplicity, here we have denoted $\mathrm{pf}(1,2,\ldots,2N)$ to be $(1,2,\ldots,2N)$ without confusion. Substituting the above Pfaffians into (2), we have

$$
\begin{aligned}
&4D_xD_t f\cdot f + y\big(D_x^4 f\cdot f + 3D_y^2 f\cdot f\big) + 2xD_xD_y f\cdot f + 4f_y f\\
&\quad = 2\{f(4f_{xt} + yf_{xxxx} + 3yf_{yy} + 2xf_{xy} + 2f_y)\\
&\qquad - f_x(4f_t + 4yf_{xxx} - 2xf_y) + 3y(f_{xx} + f_y)(f_{xx} - f_y)\}\\
&\quad = 24y\{(1,2,\ldots,2N)(1,2,\ldots,2N-2,2N+1,2N+2)\\
&\qquad - (1,2,\ldots,2N-1,2N+1)(1,2,\ldots,2N-2,2N,2N+2)\\
&\qquad + (1,2,\ldots,2N-1,2N+2)(1,2,\ldots,2N-2,2N,2N+1)\}.
\end{aligned} \tag{28}
$$

These Pfaffians no longer satisfy the bilinear equation (2). Following the Hirota–Ohta procedure, we now introduce two new variables g and h defined by

$$g = (1,2,\ldots,2N-2), \qquad h = (1,2,\ldots,2N+2). \tag{29}$$

Then through the Pfaffian identities (21) and (22), we can show that f, g, and h satisfy the three bilinear equations

$$4D_xD_t f\cdot f + y\big(D_x^4 f\cdot f + 3D_y^2 f\cdot f\big) + 2xD_xD_y f\cdot f + 4f_y f = 24ygh, \tag{30}$$

$$\big(yD_x^3 - 2D_t + 3yD_xD_y - xD_y\big)g\cdot f + 2fg_x = 0, \tag{31}$$

$$\big(yD_x^3 - 2D_t - 3yD_xD_y - xD_y\big)h\cdot f - 2fh_x = 0. \tag{32}$$

Note that the procedures for deducing equations (31) and (32) are similar to that of (30) and are omitted. We call (30)–(32) the coupled nonisospectral KP equation. The bilinear nonisospectral KP equation (2) can be considered as a reduction of (30)–(32), in fact. Setting $u = 2(\log f)_{xx}$, $g = \rho f$ and $h = \sigma f$, we can derive the following nonlinear coupled system from (30)–(32):

$$4u_t + y(u_{xxx} + 6uu_x + 3\partial^{-1}u_{yy}) + 2xu_y + 4\partial^{-1}u_y = 24(\rho\sigma)_x, \tag{33}$$

$$3yu\rho_x + (y\rho_{xxx} - 2\rho_t + 3y\rho_{xy} - x\rho_y + 2\rho_x) + \rho\int^x (3yu_y + u)\,\mathrm{d}x = 0, \tag{34}$$

$$3yu\sigma_x + (y\sigma_{xxx} - 2\sigma_t - 3y\sigma_{xy} - x\sigma_y - 2\sigma_x) - \sigma\int^x (3yu_y + u)\,\mathrm{d}x = 0. \tag{35}$$

We can see that when $\rho = \sigma = 0$, the above coupled system can be reduced to the nonisospectral KP equation (1).

3.2. Gramm-type Pfaffian solutions to the coupled nonisospectral KP equation (30)–(32)

It is well known that the coupled KP equation derived by Pfaffianization has solutions expressed in the form of Gramm-type Pfaffians. Similarly, we can expect the coupled nonisospectral KP equation (30)–(32) to possess a Gramm-type Pfaffian solution

$$f = (1,\ldots,2N), \tag{36}$$

$$g = (c_1, c_0, 1,\ldots,2N), \tag{37}$$

$$h = (d_0, d_1, 1,\ldots,2N). \tag{38}$$

Each Pfaffian element is defined by

$$(i,j) = c_{ij} + \int^x (f_i g_j - f_j g_i)\,\mathrm{d}t, \qquad c_{ij} = -c_{ji}, \tag{39}$$

$$(d_n, i) = \frac{\partial^n}{\partial x^n} f_i, \qquad (c_n, i) = \frac{\partial^n}{\partial x^n} g_i, \qquad (d_l, d_m) = (c_l, c_m) = (d_m, c_l) = 0, \tag{40}$$

where f_i and g_j satisfy the same differential relations as in (6), (7) and (8).

Based on the above Pfaffian elements and (6)–(8), we get these differential formulae:

$$f_x = (c_0, d_0, \bullet), \tag{41}$$

$$f_{xx} = (c_0, d_1, \bullet) + (c_1, d_0, \bullet), \tag{42}$$

$$f_y = (c_0, d_1, \bullet) - (c_1, d_0, \bullet), \tag{43}$$

$$f_{xxx} = (c_0, d_2, \bullet) + (c_2, d_0, \bullet) + 2(c_1, d_1, \bullet), \tag{44}$$

$$f_{xxxx} = (c_0, d_3, \bullet) + (c_3, d_0, \bullet) + 3(c_2, d_1, \bullet) + 3(c_1, d_2, \bullet) + 2(c_1, d_1, c_0, d_0, \bullet), \tag{45}$$

$$f_{yy} = (c_0, d_3, \bullet) - (c_2, d_1, \bullet) - (c_1, d_2, \bullet) + (c_3, d_0, \bullet) - 2(c_0, d_1, c_1, d_0, \bullet), \tag{46}$$

$$f_t = (-y)[(c_0, d_2, \bullet) - (c_1, d_1, \bullet) + (c_2, d_0, \bullet)] + \tfrac{1}{2}x[(c_1, d_0, \bullet) - (c_0, d_1, \bullet)]. \tag{47}$$

Here we have denoted $(1, 2, \ldots, 2N) = (\bullet)$. By employing the above Pfaffian expressions, equation (30) is reduced to the Pfaffian identity

$$\begin{aligned} 24y\{&(c_0, d_0, c_1, d_1, \bullet)(\bullet) - (c_0, d_0, \bullet)(c_1, d_1, \bullet) \\ &+ (c_0, d_1, \bullet)(c_1, d_0, \bullet) - (c_1, c_0, \bullet)(d_0, d_1, \bullet)\} \equiv 0. \end{aligned} \tag{48}$$

Similarly, equations (31)–(32) can also be reduced to Pfaffian identities. Therefore, we have proved that the coupled system of the nonisospectral KP equation has the Gramm-type Pfaffian solution given by equations (36)–(38).

4. Conclusion

In this paper, we have presented Grammian solutions to the bilinear nonisospectral KP equation (2). Then we have successfully applied Hirota–Ohta's Pfaffianization procedure to equation (2) to generate a coupled system. Similar to the coupled KP equation [2], it has been shown that the coupled system also possesses Gramm-type Pfaffian solution.

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